SOME UPPER SUM/LOWER SUM PROBLEMS

① # 9, page 261:

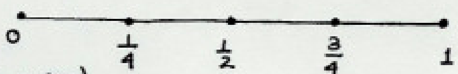
$$\left[5\left(\tfrac{1}{8}\right)+3\right]+\left[5\left(\tfrac{2}{8}\right)+3\right]+\cdots\cdots+\left[5\left(\tfrac{8}{8}\right)+3\right]=\sum_{i=1}^{8}\left[5\left(\tfrac{i}{8}\right)+3\right]$$

② # 19, page 261:

$$\sum_{i=1}^{15} i(i-1)^2=\sum_{i=1}^{15} i(i^2-2i+1)=\sum_{i=1}^{15}\left[i^3-2i^2+i\right]=\left(\sum_{i=1}^{15} i^3\right)-2\left(\sum_{i=1}^{15} i^2\right)+\left(\sum_{i=1}^{15} i\right)$$

$$=\left[\frac{15(15+1)}{2}\right]^2-2\left[\frac{15(15+1)(30+1)}{6}\right]+\left[\frac{15(15+1)}{2}\right]=120^2-2(1240)+120=12040$$

③ # 27, pg 262: $f(x)=\sqrt{x}$; $n=4$



$\Delta x=\frac{1}{4}$

Refer to the given graph of $f(x)=\sqrt{x}$ (pg 262)

$$\text{Lower Sum}=s(n)=s(4)=f(0)\cdot\Delta x+f\left(\tfrac{1}{4}\right)\Delta x+f\left(\tfrac{1}{2}\right)\Delta x+f\left(\tfrac{3}{4}\right)\cdot\Delta x$$

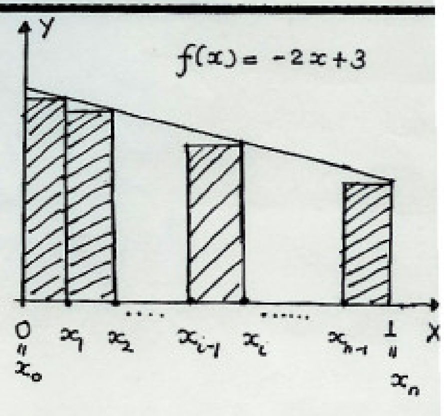
$$=(0)\cdot\frac{1}{4}+\sqrt{\frac{1}{4}}\cdot\frac{1}{4}+\sqrt{\frac{1}{2}}\cdot\frac{1}{4}+\sqrt{\frac{3}{4}}\cdot\frac{1}{4}\approx 0.52$$

$$\text{Upper Sum}=S(n)=S(4)=f\left(\tfrac{1}{4}\right)\cdot\Delta x+f\left(\tfrac{1}{2}\right)\cdot\Delta x+f\left(\tfrac{3}{4}\right)\cdot\Delta x+f(1)\cdot\Delta x$$

$$=\sqrt{\frac{1}{4}}\cdot\frac{1}{4}+\sqrt{\frac{1}{2}}\cdot\frac{1}{4}+\sqrt{\frac{3}{4}}\cdot\frac{1}{4}+\sqrt{1}\cdot\frac{1}{4}\approx 0.77.$$

The calculation means that $0.52\le$ Area of the region ≤ 0.77.

④ # 47, pg 263: $f(x)=-2x+3$ on $[0,1]$



Let us use the lower sums:

$$\Delta x=\frac{1-0}{n}=\frac{1}{n}\quad;\quad x_i=0+i\left(\frac{1}{n}\right)=\frac{i}{n}$$

$$s(n)=\sum_{i=1}^{n} f(x_i)\cdot\Delta x=\sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n}=\sum_{i=1}^{n}\left(-\frac{2i}{n}+3\right)\cdot\frac{1}{n}$$

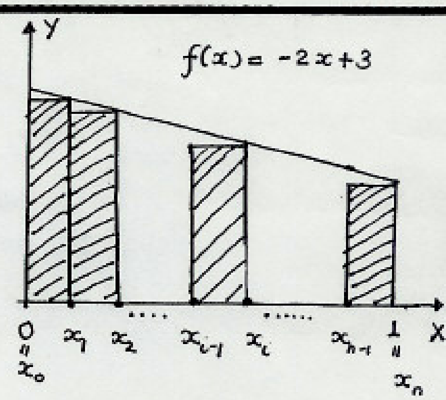
$$\therefore s(n)=\frac{1}{n}\left[-\frac{2}{n}\sum_{i=1}^{n} i+3\sum_{i=1}^{n} 1\right]$$

$$s(n)=\frac{1}{n}\left[-\frac{\cancel{2}}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}}+3\cdot n\right]=\frac{1}{n}\left[-n-1+3n\right]=\frac{1}{n}(2n-1)=2-\frac{1}{n}$$

$$\therefore \lim_{n\to\infty} s(n)=\lim_{n\to\infty}\left(2-\frac{1}{n}\right)=2-0=2$$

$\therefore$ Area of the required region $=2$ units

④ # 47, pg 263 : $f(x) = -2x+3$ on $[0,1]$



Let us use the lower sums:

$$\Delta x = \frac{1-0}{n} = \frac{1}{n} \quad ; \quad x_i = 0 + i\left(\frac{1}{n}\right) = \frac{i}{n}$$

$$s(n) = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{-2i}{n}+3\right)\cdot\frac{1}{n}$$

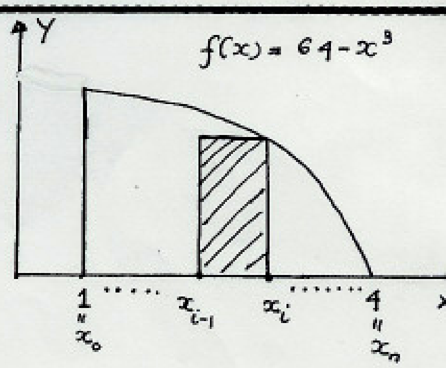
$$\therefore s(n) = \frac{1}{n}\left[-\frac{2}{n}\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} 1\right]$$

$$s(n) = \frac{1}{n}\left[\frac{-\cancel{2}}{\cancel{n}}\cdot\frac{\cancel{n}(n+1)}{\cancel{2}} + 3\cdot n\right] = \frac{1}{n}\left[-n-1+3n\right] = \frac{1}{n}(2n-1) = 2-\frac{1}{n}$$

$$\therefore \lim_{n\to\infty} s(n) = \lim_{n\to\infty}\left(2-\frac{1}{n}\right) = 2-0 = 2$$

$\therefore$ Area of the required region = 2 units

⑤ # 53, pg 263 : $f(x) = 64 - x^3$ on $[1, 4]$



$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \quad ; \quad x_i = 1 + i\cdot\Delta x = 1 + i\left(\frac{3}{n}\right) = 1 + \frac{3i}{n}$$

Again, use the lower sums:

$$s(n) = \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\cdot\frac{3}{n} = \sum_{i=1}^{n}\left[64-\left(1+\frac{3i}{n}\right)^3\right]\cdot\frac{3}{n}$$

$$= \frac{3}{n}\sum_{i=1}^{n}\left[64-1-\frac{9i}{n}-\frac{27i^2}{n^2}-\frac{27i^3}{n^3}\right]$$

$$\therefore s(n) = \frac{3}{n}\left[\sum_{i=1}^{n} 63 - \frac{9}{n}\sum_{i=1}^{n} i - \frac{27}{n^2}\sum_{i=1}^{n} i^2 - \frac{27}{n^3}\sum_{i=1}^{n} i^3\right]$$

$$= \frac{3}{n}\left[63n - \frac{9}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} - \frac{27}{n^{\cancel{2}}}\frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{27}{n^{\cancel{3}}}\frac{\cancel{n^2}(n+1)^2}{4}\right]$$

$$\therefore s(n) = 3\left[63-\frac{9}{2}\left(1+\frac{1}{n}\right)-\frac{9}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{27}{4}\left(1+\frac{1}{n}\right)^2\right]$$

$$\therefore \lim_{n\to\infty} s(n) = \lim_{n\to\infty} 3\left[63-\frac{9}{2}\left(1+\underset{\downarrow 0}{\frac{1}{n}}\right)-\frac{9}{2}\left(1+\underset{\downarrow 0}{\frac{1}{n}}\right)\left(2+\underset{\downarrow 0}{\frac{1}{n}}\right)-\frac{27}{4}\left(1+\underset{\downarrow 0}{\frac{1}{n}}\right)^2\right] = 3\left[63-\frac{9}{2}-\frac{9}{2}(1)(2)-\frac{27}{4}(1)\right]$$

$$= 513/4$$

$\therefore$ Area of the required region = 513/4 units^2 //