ANSWERS TO SOME EXERCISES:

① Given $\underline{F}(x,y,z) = \underbrace{\sin x}_{M}\,\underline{i} + \underbrace{\cos y}_{N}\,\underline{j} + \underbrace{z^2}_{P}\,\underline{k}$

Find $\mathrm{div}(\underline{F})$

$$\mathrm{div}(\underline{F}) = \nabla \cdot \underline{F} = \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right)\left(M\underline{i} + N\underline{j} + P\underline{k}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$\therefore \mathrm{div}(\underline{F}) = \cos x - \sin y + 2z$ (The answer is a scalar)

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② Given: $\underline{F}(x,y,z) = \ln(xyz)\,(\underline{i} + \underline{j} + \underline{k})$

Find $\mathrm{div}\,\underline{F}\big|_{(3,2,1)}$

$$\therefore \underline{F}(x,y,z) = \underbrace{\ln(xyz)}_{M}\,\underline{i} + \underbrace{\ln(xyz)}_{N}\,\underline{j} + \underbrace{\ln(xyz)}_{P}\,\underline{k}$$

$$\therefore \mathrm{div}(\underline{F}) = \frac{\partial M}{\partial x} + \frac{\partial M}{\partial y} + \frac{\partial N}{\partial z}$$

$$= \frac{1}{xyz}\cdot yz + \frac{1}{xyz}\cdot xz + \frac{1}{xyz}\cdot xy$$

$$\therefore \mathrm{div}(\underline{F}(x,y,z)) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\therefore \mathrm{div}\,\underline{F}(3,2,1) = \frac{1}{3} + \frac{1}{2} + \frac{1}{1} = \frac{11}{6}.$$

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③ Given: $\underline{F}(x,y,z) = \underline{i} + 2x\,\underline{j} + 3y\,\underline{k}$

$\underline{G}(x,y,z) = x\,\underline{i} - y\,\underline{j} + z\,\underline{k}$

(a) $\mathrm{div}(\underline{F}) = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z} = 0 + 0 + 0 = 0$

(Here $M = 1$, $N = 2x$, $P = 3y$)

(b) $\text{Curl}(\underline{G}) = \underline{\nabla} \times \underline{G}$

$$= \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) \times (M\underline{i} + N\underline{j} + P\underline{k})$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) - \underline{j}\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right) + \underline{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

use this with $M = x$; $N = -y$; $P = z$

$$= \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(0-0)$$

$\therefore\ \text{Curl}(\underline{G}) = \underline{0}$ (This is a vector!)
i.e. zero vector

(c) $\text{div}(\underline{F} \times \underline{G})$

First: $\underline{F} \times \underline{G} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2x & 3y \\ x & -y & z \end{vmatrix}$

$$\therefore\ \underline{F} \times \underline{G} = \underline{i}\,\underbrace{(2xz + 3y^2)}_{M} - \underline{j}\,(\underbrace{z - 3xy}) + \underline{k}\,\underbrace{(-y - 2x^2)}_{P}$$

$N = -(z - 3xy)$

$$\therefore\ \text{div}(\underline{F} \times \underline{G}) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

$$= (2z) + (3x) + (0)$$

$$\therefore\ \text{div}(\underline{F} \times \underline{G}) = 3x + 2z$$