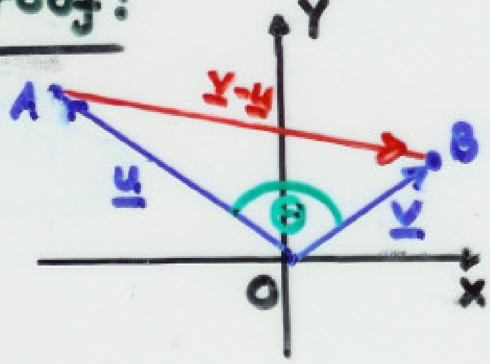
THEOREM (ANGLE BETWEEN TWO VECTORS)

Suppose that $\underline{u}$ and $\underline{v}$ are any two nonzero vectors in 2D or 3D. Let θ be the angle between them, $0 \leq \theta \leq \pi$. Then $\underline{u} \cdot \underline{v} = \|\underline{u}\| \, \|\underline{v}\| \cos\theta$, or equivalently $\cos\theta = \dfrac{\underline{u} \cdot \underline{v}}{\|\underline{u}\| \, \|\underline{v}\|}$

<u>proof:</u>



Let's represent $\underline{u}$ by $\overrightarrow{OA}$, and $\underline{v}$ by $\overrightarrow{OB}$. Then $\overrightarrow{AB}$ represents the vector $\underline{v} - \underline{u}$ (Know why?)
Use the Law of Cosines for the Δ OAB.

$$\therefore \|\underline{v}-\underline{u}\|^2 = \|\underline{u}\|^2 + \|\underline{v}\|^2 - 2\|\underline{u}\|\,\|\underline{v}\| \cos\theta \qquad (1)$$

Now simplify the L.H.S using properties of dot pdt:

$$\|\underline{v}-\underline{u}\|^2 = (\underline{v}-\underline{u}) \cdot (\underline{v}-\underline{u}) \qquad (\because \|\underline{w}\|^2 = \underline{w} \cdot \underline{w} \text{ for any vector } \underline{w})$$
$$= (\underline{v}-\underline{u}) \cdot \underline{v} - (\underline{v}-\underline{u}) \cdot \underline{u} \qquad (\text{by Distributive Prop.})$$
$$= \underline{v} \cdot \underline{v} - \underline{u} \cdot \underline{v} - \underline{v} \cdot \underline{u} + \underline{u} \cdot \underline{u} \qquad (\text{by Distributive Prop})$$

$$\therefore \|\underline{v}-u\|^2 = \|\underline{v}\|^2 - 2\,\underline{u} \cdot \underline{v} + \|\underline{u}\|^2 \qquad \left(\because \underline{u} \cdot \underline{v} = \underline{v} \cdot \underline{u} \text{ and } \underline{u} \cdot \underline{u} = \|\underline{u}\|^2, \ \underline{v} \cdot \underline{v} = \|\underline{v}\|^2\right.$$

plug this back in the L.H.S of (1):

$$\therefore \cancel{\|\underline{v}\|^2} - 2\,\underline{u} \cdot \underline{v} + \cancel{\|\underline{u}\|^2} = \cancel{\|\underline{u}\|^2} + \cancel{\|\underline{v}\|^2} - 2\|\underline{u}\|\,\|\underline{v}\| \cos\theta$$

$$\therefore \underline{u} \cdot \underline{v} = \|\underline{u}\|\,\|\underline{v}\| \cos\theta \quad \text{or} \quad \cos\theta = \frac{\underline{u} \cdot \underline{v}}{\|\underline{u}\|\,\|\underline{v}\|} \qquad \text{Q.E.D.}$$