Projection of a given vector onto another vector

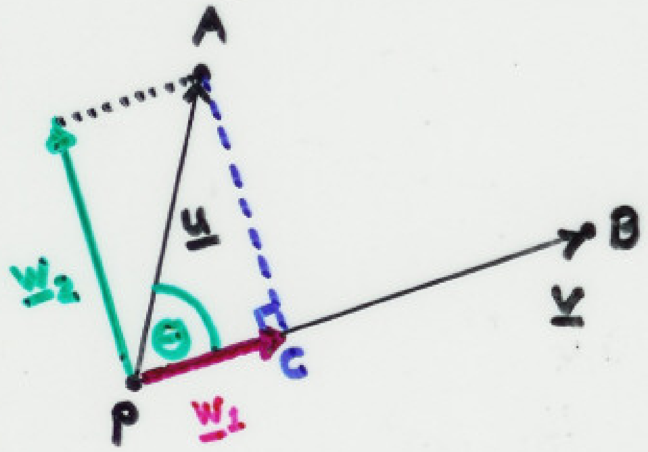
Suppose $\underline{u}$ and $\underline{v}$ are two given nonzero vectors.

Q: What is meant by the projection of $\underline{u}$ onto $\underline{v}$?

This will be denoted by $\text{Proj}_{\underline{v}}\,\underline{u}$

A: The idea is really simple! The Key is to think geometrically.

We will illustrate it for the case where the angle θ between $\underline{u}$ and $\underline{v}$ are acute.



Draw a perpendicular line from P to line PB
(we say "project" point A to PB)
In this way, obtain point C
Then the vector $\overrightarrow{PC}$ is called the proj. of $\underline{u}$ onto $\underline{v}$

So how do we calculate $\overrightarrow{PC}$?

First, the magnitude of $\overrightarrow{PC}$ can be calculated using the right Δ PAC.

$$\cos\theta = \frac{\|\overrightarrow{PC}\|}{\|\underline{u}\|}, \quad \text{so} \quad \|\overrightarrow{PC}\| = \|\underline{u}\|\cos\theta \qquad \text{—— (1)}$$

Secondly, the direction of $\overrightarrow{PC}$ is the same as that of $\underline{v}$
we know the unit vector in the dir. of $\underline{v}$ is $\dfrac{\underline{v}}{\|\underline{v}\|}$ —— (2)

$\therefore$ By ① & ② $\quad \overrightarrow{PC} = \|\underline{u}\| \cos\theta \left(\dfrac{\underline{v}}{\|\underline{v}\|}\right)$

i.e. $\quad \text{Proj}_{\underline{v}}\,\underline{u} = \|\underline{u}\| \cos\theta \left(\dfrac{\underline{v}}{\|\underline{v}\|}\right)$ ——— ③

But since $\cos\theta = \dfrac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\|\underline{v}\|}$, we can rewrite ③:

$$\therefore \text{Proj}_{\underline{v}}\,\underline{u} = \cancel{\|\underline{u}\|}\, \frac{\underline{u}\cdot\underline{v}}{\cancel{\|\underline{u}\|}\,\|\underline{v}\|}\left(\frac{\underline{v}}{\|\underline{v}\|}\right) = \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\,\underline{v}$$

$$\boxed{\therefore \quad \text{Proj}_{\underline{v}}\,\underline{u} = \left(\frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\right)\underline{v}}$$

NOTE ① $\text{Proj}_{\underline{v}}\,\underline{u}$ is also called the vector component of $\underline{u}$ along $\underline{v}$, and also denoted by $\underline{w}_1$

i.e. $\quad \underline{w}_1 = \text{Proj}_{\underline{v}}\,\underline{u} = \left(\dfrac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}\right)\underline{v}$

② Similarly, the green vector $\underline{w}_2$ in the diagram is called the vector component of $\underline{u}$ orthogonal to $\underline{v}$.

③ By Parallelogram Rule, always $\underline{u} = \underline{w}_1 + \underline{w}_2$