Example: Let G, H be any 2 groups & let $G \times H$ be their DIRECT PRODUCT

Let $S = \{(x, e_H) \mid x \in G\}$

Show that S is a subgroup of $G \times H$.

Proof:

$S \neq \Phi$ because $(e_G, e_H) \in S$.

Also, clearly, $S \subseteq G \times H$

Let $a, b \in S$ be arbitrary.

Want to show that $ab^{-1} \in S$.

Since $a \in S$, by defn, $a = (x, e_H)$ for some $x \in G$

Since $b \in S$, by defn, $b = (y, e_H)$ for some $y \in G$

$$\begin{aligned} \therefore\ ab^{-1} &= (x, e_H) \cdot (y, e_H)^{-1} \\ &= (x, e_H) \cdot (y^{-1}, e_H)^{-1} \\ &= (x, e_H) \cdot (y^{-1}, e_H) \\ &= (xy^{-1}, e_H \cdot e_H) \\ &= (xy^{-1}, e_H) \end{aligned}$$

$\therefore\ ab^{-1} = (xy^{-1}, e_H) \in S$

$\therefore$ By defn of a subgroup, $S \leq G \times H$ Q.E.D