Example: Let G be an abelian group.
Let H be a subgroup of G.
Let $K = \{x \in G \mid x^2 \in H\}$
Prove that K is a subgroup of G.

Proof:

$\Phi \neq K$ because $e \in K$.
$K \subseteq G$ by definition of K.

Let $a, b \in K$ be arbitrary.
Want to show that $ab^{-1} \in K$.
[This is the same as showing $ab^{-1} \in G$ & $(ab^{-1})^2 \in H$]

Since $a \in K$, by defn, $a \in G$ and $a^2 \in H$
Since $b \in K$, by defn, $b \in G$ and $b^2 \in H$
$\therefore \; ab^{-1} \in G$
Also $(ab^{-1})^2 = a^2(b^{-1})^2$
$= a^2(b^2)^{-1}$
$\therefore (ab^{-1})^2 = a^2(b^2)^{-1} \in H$
Now, $ab^{-1} \in G$ and $(ab^{-1})^2 \in H$
$\therefore$ By defn of K, $ab^{-1} \in K$
$\therefore K$ is a subgroup of G. Q.E.D.