D3 (Pg 143) Show that the center of any gp G is a normal subgroup of G.

Proof: Let G be any group.

Recall that the center of G is given by,

$$Z(G) = \{a \in G \mid ax = xa \ \forall x \in G\}$$

Earlier we have shown that $Z(G)$ is a subgroup of G. Review the proof and write it again here.

Now to show the 'normality', let $a \in Z(G)$ and $x \in G$ be arbit.

[Need to show that $xax^{-1} \in Z(G)$]

In order to show that $xax^{-1} \in Z(G)$, we need to show that xax^{-1} must commute with every element of G.

So, let $y \in G$ be arbitrary, and show that $y.(xax^{-1}) = (xax^{-1})y$.

$$
\begin{aligned}
L.H.S = y(xax^{-1}) &= y(xa)(x^{-1}) \\
&= y(ax)(x^{-1}) && (\because \text{ since } a \in Z(G),\ ax = xa) \\
&= ya(xx^{-1}) && (\because \text{ associative property}) \\
&= ya && (\because xx^{-1} = e \text{ and group axioms})
\end{aligned}
$$

$$
\begin{aligned}
R.H.S = (xax^{-1})y &= x(ax^{-1})y \\
&= x(x^{-1}a)y && (\because \text{ Since } a \in Z(G),\ ax^{-1} = x^{-1}a) \\
&= (xx^{-1})(ay) && (\because \text{ asso. prop}) \\
&= ay \\
&= ya && (\because \text{ since } a \in Z(G),\ ay = ya)
\end{aligned}
$$

$\therefore$ L.H.S = R.H.S i.e. $y(xax^{-1}) = (xax^{-1})y$ where $y \in G$ is arbi

Also, clearly $xax^{-1} \in G$ ($\because a, x \in G$ & by gp. axioms)

$\therefore$ By defn, $xax^{-1} \in Z(G)$

$\therefore Z(G) \trianglelefteq G$

Q.E.D