THE CENTER OF A GROUP

Let G be any group. Then we will define the Center of G, denoted by $Z(G)$ as follows :

$$Z(G) = \{a \in G \mid ax = xa \ \forall x \in G\}$$

In otherwords, $Z(G)$ consists of all elements of G, that commutes with every element of G.

Then $Z(G)$ is a subgroup of G.

Proof:

First $e \in Z(G)$ because e (id elmt) commute with any elmt of G. This is so because $ex = xe \ (= x) \ \forall x \in G$.

$$\therefore \quad \Phi \neq Z(G)$$

Clearly, by defn of $Z(G)$, $Z(G) \subseteq G$

$$\therefore \quad \Phi \neq Z(G) \subseteq G$$

Now let $a, b \in Z(G)$ be arbitrary

Want to show $ab^{-1} \in Z(G)$

Since $a \in Z(G)$, $ax = xa \ \forall x \in G$ ——— (α)

Since $b \in Z(G)$, $bx = xb \ \forall x \in G$ ——— (β)

To show $ab^{-1} \in Z(G)$, NEED TO SHOW THAT $(ab^{-1})(x) = x(ab^{-1}) \ \forall x \in G$.

So, let $x \in G$ be arbitrary

$$\begin{aligned}
\therefore (ab^{-1})(x) &= a(b^{-1}x) \\
&= a(xb^{-1}) \quad (\because \text{by } (\beta),\ b^{-1}x = xb^{-1}) \\
&= (ax)b^{-1} \\
&= (xa)b^{-1} \quad (\because \text{by } (\alpha),\ ax = xa) \\
&= x(ab^{-1})
\end{aligned}$$

$\therefore (ab^{-1})x = x(ab^{-1}) \ \forall x \in G$

Also $ab^{-1} \in G$ clearly.

$\therefore$ By definition of $Z(G)$, $ab^{-1} \in Z(G)$

$\therefore \forall a, b \in Z(G)$, $ab^{-1} \in Z(G)$

$\therefore$ By definition of a subgroup, $Z(G) \leq G$

Q.E.D