The Centralizer of an element in a group

Defn: Let G be any group and $a \in G$.
Then then centralizer of a in G is defined as $C(a) = \{x \in G \mid ax = xa\}$

THEOREM: $C(a)$ is a subgroup of G

Proof:

$\Phi \neq C(a)$ because $e \in C(a)$

$C(a) \subseteq G$, because by defn of $C(G)$

Let $x, y \in C(a)$ be arbitrary.

Want to show that $xy^{-1} \in C(a)$

[This is the same as showing $xy^{-1} \in G$ and $a(xy^{-1}) = (xy^{-1})a$]

Since $x \in C(a)$, $x \in G$ and $ax = xa$ —— (i)

Since $y \in C(a)$, $y \in G$ and $ay = ya$ —— (ii)

$\therefore \quad xy^{-1} \in G$

$$\begin{aligned} \text{Also} \quad a(xy^{-1}) &= (ax)y^{-1} \\ &= x(ay^{-1}) \\ &= (xy^{-1})a \end{aligned}$$

$\therefore \; a(xy^{-1}) = (xy^{-1})a$

Now $xy^{-1} \in G$ and $a(xy^{-1}) = (xy^{-1})a$

$\therefore$ By defn of $C(a)$, $xy^{-1} \in C(a)$

$\therefore$ $C(a)$ is a subgroup of G.

Q.E.D.

<u>NOTE</u>: For a given $a \in G$, $C(a)$ has some interesting properties.

[See pg 55, Abs Alg by I.N. Herstein]